Conditional *q*-entropies and quantum separability: a numerical exploration

Conditional *q*-entropies and quantum separability: a numerical exploration

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Abstract
We revisit the relationship between quantum separability and the sign of the
relative *q*-entropies of composite quantum systems. The *q*-entropies depend on
the density matrix eigenvalues p_i through the quantity $\omega_q = \sum_i p_i^q$. Rényi's
and Tsallis' measures constitute particular instances of these entropies. We
perform a systematic numerical survey of the space of mixed states of two-
qubit systems in order to determine, as a function of the degree of mixture,
and for different values of the entropic parameter *q*, the volume in state space
occupied by those states characterized by positive values of the relative entropy.
Similar calculations are performed for qubit–qutrit systems and for composite
systems described by Hilbert spaces of larger dimensionality. We pay particular
attention to the limit case $q \to \infty$. Our numerical results indicate that, as the
dimensionalities of both subsystems increase, typical mixed states of composite
quantum systems tend, as far as their relative *q*-entropies are concerned, to
behave in a classical way.

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1. Introduction

Important steps have been made recently towards a systematic exploration of the space of
arbitrary (pure or mixed) states of composite quantum systems [1–4] in order to determine
the typical features exhibited by these states with regard to the phenomenon of quantum
entanglement [1–3, 5–8]. This phenomenon is one of the most fundamental and non-classical
features exhibited by quantum systems [9]. Quantum entanglement lies at the basis of some of
the most important processes studied by quantum information theory [9–17], such as quantum
cryptographic key distribution [18], quantum teleportation [19], superdense coding [20] and
quantum computation [21, 22]. A state of a composite quantum system is called 'entangled' if
it cannot be represented as a mixture of factorizable pure states. Otherwise, the state is called

separable. The above definition is physically meaningful because entangled states (unlike separable states) cannot be prepared locally by acting on each subsystem individually [23–25].

When one deals with a classical composite system, described by a suitable probability distribution defined over the concomitant phase space, the entropy of any of its subsystems is always equal to or smaller than the entropy characterizing the whole system. This is also the case for separable states of a composite quantum system [26, 27]. In contrast, a subsystem of a quantum system described by an entangled state may have an entropy greater than the entropy of the whole system. In fact, the von Neumann entropy of either of the subsystems of a bipartite quantum system described (as a whole) by a pure state provides a natural measure of the amount of entanglement of such state. Thus, a pure state (which has vanishing entropy) is entangled if and only if its subsystems have an entropy larger than that associated with the system as a whole. The situation is more complex when the composite system is described by a mixed state. As already mentioned, there are entangled mixed states such that the entropy of the complete system is smaller than the entropy of one of its subsystems. Alas, entangled mixed states such that the entropy of the system as a whole is larger than the entropy of either of its subsystems exist as well. Consequently, the classical inequalities relating the entropy of the whole system with the entropies of its subsystems provide only necessary, but not sufficient, conditions for quantum separability. There are several entropic (or information) measures that can be used in order to implement these entropic criteria for separability. Considerable attention has been paid, in this regard, to the q-entropies [25, 27–34], which incorporate both Rényi's [35] and Tsallis' [36–39] families of information measures as special instances (both admitting, in turn, Shanon's measure as the particular case associated with the limit $q \to 1$). The early motivation for these studies was the development of practical separability criteria for density matrices. The discovery by Peres of the partial transpose criteria, which for two-qubit and qubit–qutrit systems turned out to be both necessary and sufficient, rendered that original motivation somewhat outmoded. In fact, it is not possible to find a necessary and sufficient criterion for separability based solely upon the eigenvalue spectra of the three density matrices ρ_{AB}, $\rho_A = \mathrm{Tr}_B[\rho_{AB}]$ and $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$ associated with a composite system $A \otimes B$ [26]. However, *the violation of classical entropic inequalities by entangled quantum states is of considerable interest in its own right*. Quantum entanglement is a fundamental aspect of quantum physics that deserves to be investigated in full detail from all possible points of view. The violation of the classical entropic inequalities provides a clear and direct information-theoretical manifestation of the phenomenon of entanglement.

The aim of the present work is to study the relationship between quantum separability and the violation of the classical q-entropic inequalities (which corresponds to negative values of the relative q-entropies). We will perform a systematic numerical survey of the space of mixed states of two-qubit systems in order to determine, as a function of the degree of mixture, and for different values of the entropic parameter q, the volume in state space occupied by those states characterized by positive values of the relative q-entropies. Similar calculations are performed for qubit–qutrit systems and for composite systems described by Hilbert spaces of larger dimensionality. We pay particular attention to the limit case $q \to \infty$.

The paper is organized as follows. In section 2 we review some basic properties of the q-entropies and the relative q-entropies. Our main results are discussed in sections 3. Finally, some conclusions are drawn in section 4.

2. q-entropies and q-relative entropies

There are several entropic (or information) measures that can be useful in order to investigate the violation of classical entropic inequalities by quantum entangled states. The von Neumann

measure

$$S_1 = -\text{Tr}(\hat{\rho} \ln \hat{\rho}) \tag{1}$$

is important because of its relationship with the thermodynamic entropy. On the other hand, the so-called participation ratio

$$R(\hat{\rho}) = \frac{1}{\text{Tr}(\hat{\rho}^2)} \tag{2}$$

is particularly convenient for calculations [1, 5, 40]. It varies from unity for pure states to N for totally mixed states (if $\hat{\rho}$ is represented by an $N \times N$ matrix). It may be interpreted as the effective number of pure states that enter the mixture. If the participation ratio of $\hat{\rho}$ is high enough, then its partially transposed density matrix is positive, which for $N = 4$ amounts to separability. The q-entropies, which are functions of the quantity

$$\omega_q = \text{Tr}(\hat{\rho}^q) \tag{3}$$

provide one with a whole family of entropic measures. In the limit $q \rightarrow 1$ these measures incorporate (1) as a particular instance. On the other hand, when $q = 2$ they are simply related to the participation ratio (2). Most of the applications of q-entropies to physics involve either the Rényi entropies [35],

$$S_q^{(R)} = \frac{1}{1-q} \ln(\omega_q) \tag{4}$$

or the Tsallis entropies [36–38]

$$S_q^{(T)} = \frac{1}{q-1}(1 - \omega_q). \tag{5}$$

We reiterate that the von Neumann measure (1) constitutes a particular instance of both Rényi's and Tsallis' entropies, which is obtained in the limit $q \rightarrow 1$. The most distinctive single property of Tsallis' entropy is its nonextensivity. Tsallis' entropy of a composite system $A \otimes B$ whose state is described by a factorizable density matrix, $\rho_{AB} = \rho_A \otimes \rho_B$, is given by Tsallis' q-additivity law

$$S_q^{(T)}(\rho_{AB}) = S_q^{(T)}(\rho_A) + S_q^{(T)}(\rho_B) + (1-q)S_q^{(T)}(\rho_A)S_q^{(T)}(\rho_B). \tag{6}$$

In contrast, Rényi's entropy is extensive. That is, if $\rho_{AB} = \rho_A \otimes \rho_B$,

$$S_q^{(R)}(\rho_{AB}) = S_q^{(R)}(\rho_A) + S_q^{(R)}(\rho_B). \tag{7}$$

Tsallis' and Rényi's measures are related through

$$S_q^{(T)} = F\left(S_q^{(R)}\right) \tag{8}$$

where the function F is given by

$$F(x) = \frac{1}{1-q}\{e^{(1-q)x} - 1\}. \tag{9}$$

An immediate consequence of equations (8) and (9) is that, for all non-vanishing values of q, Tsallis' measure $S_q^{(T)}$ is a monotonic increasing function of Rényi's measure $S_q^{(R)}$.

Considerable attention has been recently paid to a relative entropic measure based upon Tsallis' functional, and defined as

$$S_q^{(T)}(A|B) = \frac{S_q^{(T)}(\rho_{AB}) - S_q^{(T)}(\rho_B)}{1 + (1-q)S_q^{(T)}(\rho_B)}. \tag{10}$$

Here ρ_{AB} designs an arbitrary quantum state of the composite system $A \otimes B$, not necessarily factorizable nor separable, and $\rho_B = \text{Tr}_A(\rho_{AB})$. The relative q-entropy $S_q^{(T)}(B|A)$ is defined

in a similar way as (10), replacing ρ_B by $\rho_A = \text{Tr}_B(\rho_{AB})$. The relative q-entropy (10) has been recently studied in connection with the separability of density matrices describing composite quantum systems [32, 33]. For separable states, we have [27]

$$S_q^{(T)}(A|B) \geqslant 0 \qquad S_q^{(T)}(B|A) \geqslant 0. \tag{11}$$

In contrast, there are entangled states that have negative relative q-entropies. That is, for some entangled states one (or both) of the inequalities (11) are not verified.

Note that the denominator in (10),

$$1 + (1 - q)S_q^{(T)} = w_q > 0 \tag{12}$$

is always positive. Consequently, as far as the sign of the relative entropy is concerned, the denominator in (10) can be ignored. Besides, since Tsallis' entropy is a monotonic increasing function of Rényi's (see equations (8) and (9)), it is plain that (10) has always the same sign as

$$S_q^{(R)}(A|B) = S_q^{(R)}(\rho_{AB}) - S_q^{(R)}(\rho_B). \tag{13}$$

From now on we refer to the positivity of either Tsallis' relative entropy (10) or Rényi's relative entropy (13) as the 'classical q-entropic inequalities'. In general, when we speak about the sign of the q-relative entropy, we refer indistinctly either to the sign of (10) or to the sign of (13) (which always coincides).

3. Probabilities of finding states with positive relative q-entropies

In order to perform a systematic numerical survey of the properties of arbitrary (pure and mixed) states of a given quantum system, it is necessary to introduce an appropriate measure μ on the concomitant space $\mathcal{S}$ of general quantum states. Such a measure is needed to compute volumes within the space $\mathcal{S}$, as well as to determine what is to be understood by a uniform distribution of states on $\mathcal{S}$. A natural measure on $\mathcal{S}$, which we are going to adopt in the present work, was recently introduced by Zyczkowski *et al* [1, 2]. An arbitrary (pure or mixed) state ρ of a quantum system described by an N-dimensional Hilbert space can always be expressed as the product of three matrices,

$$\rho = U D[\{\lambda_i\}] U^\dagger. \tag{14}$$

Here U is an $N \times N$ unitary matrix and $D[\{\lambda_i\}]$ is an $N \times N$ diagonal matrix whose diagonal elements are $\{\lambda_1, \ldots, \lambda_N\}$, with $0 \leqslant \lambda_i \leqslant 1$, and $\sum_i \lambda_i = 1$. The group of unitary matrices $U(N)$ is endowed with a unique, uniform measure: the Haar measure ν [41]. On the other hand, the N-simplex Δ, consisting of all the real N-uples $\{\lambda_1, \ldots, \lambda_N\}$ appearing in (14), is a subset of an $(N-1)$-dimensional hyperplane of $\mathcal{R}^N$. Consequently, the standard normalized Lebesgue measure $\mathcal{L}_{N-1}$ on $\mathcal{R}^{N-1}$ provides a natural measure for Δ. The aforementioned measures on $U(N)$ and Δ lead then to a natural measure μ on the set $\mathcal{S}$ of all the states of our quantum system [1, 2, 41], namely

$$\mu = \nu \times \mathcal{L}_{N-1}. \tag{15}$$

All our present considerations are based on the assumption that the uniform distribution of states of a quantum system is the one determined by the measure (15). Thus, in our numerical computations we randomly generate states according to the measure (15).

The simplest quantum mechanical systems exhibiting the phenomenon of entanglement are the two-qubit systems ($N = 4$). They play a fundamental role in quantum information theory. The concomitant space of mixed states is 15-dimensional and its properties are not trivial. There still are features of this state space, related to the phenomenon of entanglement, which have not, thus far, been completely characterized in full detail.
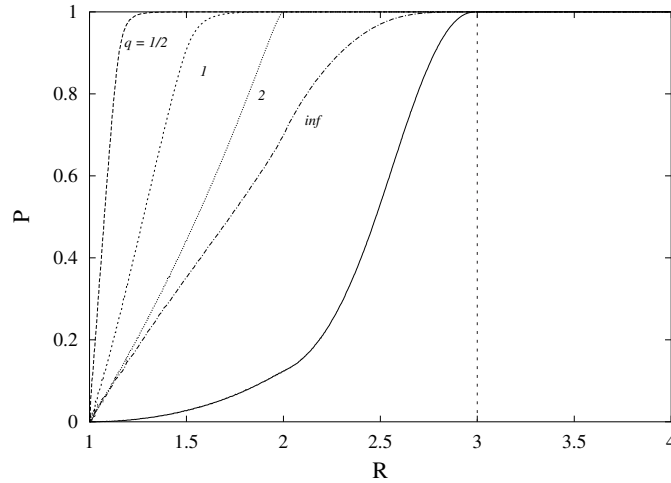
Figure 1. Probability of finding (for different values of q) a two-qubit state which, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, has positive relative q-entropies. The solid line corresponds to the probability of finding, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, a two-qubit state with a positive partial transpose.

We determined numerically, by recourse to a Monte Carlo calculation and for different values of the entropic parameter q, the probability of finding a two-qubit state which, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, has positive relative q-entropies (i.e. $S_q^{(R)}(\rho_{AB}) \geqslant S_q^{(R)}(\rho_A)$ and $S_q^{(R)}(\rho_{AB}) \geqslant S_q^{(R)}(\rho_B)$). The results are depicted in figure 1. The curve associated with the limit case $q \to \infty$ deserves special comment. In this limit we have

$$\lim_{q \to \infty} (\text{Tr}\, \rho^q)^{1/q} = \lim_{q \to \infty} \left(\sum_i p_i^q \right)^{1/q} = \lambda_m \tag{16}$$

where

$$\lambda_m = \max_i \{p_i\} \tag{17}$$

is the maximum eigenvalue of the statistical operator ρ. Hence, in the limit $q \to \infty$, the q-entropies depend only on the largest eigenvalue of the density matrix. In particular, the Rényi entropy reduces to

$$S_\infty^{(R)} = -\ln(\lambda_m). \tag{18}$$

This means that the curve in figure 1 associated with $q = \infty$ indicates the probabilities of finding states such that the largest eigenvalue of the statistical operator describing the composite system is smaller than the largest eigenvalues of either of its subsystems. The solid line in figure 1 corresponds to the probability of finding, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, a two-qubit state with a positive partial transpose. Since Peres' criterion for separability is necessary and sufficient, this last probability coincides with the probability of finding a separable state. We see that, as the value of q increases, the curves associated with the relative entropies approach the curve corresponding to the Peres criterion. However, even in the limit $q \to \infty$ the entropic curve lies above the Peres criterion by a considerable amount. This means that, even for $q \to \infty$, *there is a considerable volume in state space occupied by entangled states complying with the classical entropic inequalities* (that is, having positive relative entropies).
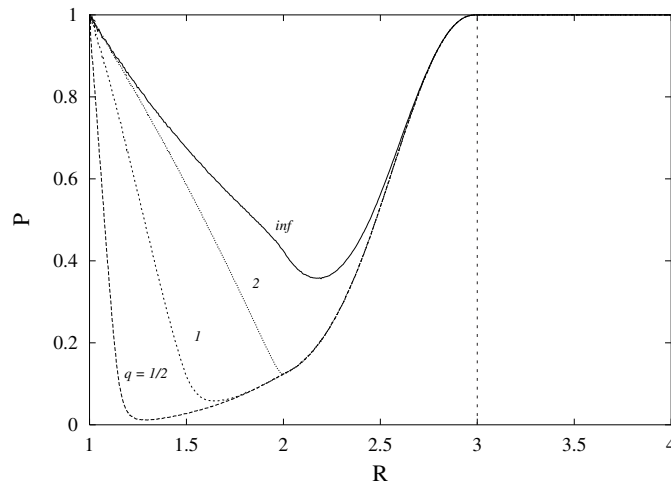
Figure 2. Probability of finding (for different values of q) a two-qubit state which, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, either (i) has both relative q-entropies positive, as well as a positive partial transpose, or (ii) has a negative relative q-entropy and a non-positive partial transpose.

The probability of finding separable states increases with the degree of mixture [1], as is evident from the solid curve in figure 1. Also, one can appreciate the fact that a similar trend is exhibited by the probability of finding, for a given q-value, states with positive relative q-entropies.

We have computed numerically the probability (for different values of q) that a two-qubit state with a given degree of mixture be correctly classified, either as entangled or as separable, on the basis of the sign of the relative q-entropies. The results are plotted in figure 2. That is, figure 2 depicts the probability of finding (for different values of q) a two-qubit state which, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, either has (i) both relative q-entropies positive, as well as a positive partial transpose, or (ii) has a negative relative q-entropy and a non-positive partial transpose. We see that, for all values of $q > 0$, this probability is equal to 1 both for pure states ($R = 1$) and for states with ($R > 3$). The probability attains its lowest value $P_m(q)$ at a special value $R_m(q)$ of the participation ratio. Both quantities $R_m(q)$ and $P_m(q)$ exhibit a monotonic increasing behaviour with q, adopting their maximum values in the limit $q \to \infty$.

In figures 1 and 2 we have used the participation ratio R as a measure of mixedness. The quantity R is, essentially, a q-entropy with $q = 2$. The q-entropies associated with other values of q are legitimate measures of mixedness as well, and have already found applications in relation with the study of entanglement [1, 8]. It is interesting to see what happens, in the present context, when we consider measures of mixedness based on other values of q. Of particular interest is the limit case $q \to \infty$ which, as already mentioned, is related to the largest eigenvalue of the density matrix. The largest eigenvalue constitutes a legitimate measure of mixture in its own right: states with larger values of λ_m can be regarded as less mixed. Its extreme values correspond to (i) pure states (with $\lambda_m = 1$) and (ii) the density matrix $\frac{1}{4}\hat{I}$ (with $\lambda_m = 1/4$). In figures 3 and 4 we have considered (in the horizontal axes) the largest eigenvalue λ_m as a measure of mixedness. We computed the probability of finding (for different values of q) a two-qubit state which, for a given value of the maximum eigenvalue λ_m, has positive relative q-entropies. The results are depicted in figure 3. The solid line corresponds to the probability of finding, for a given value of λ_m, a two-qubit state with a positive partial
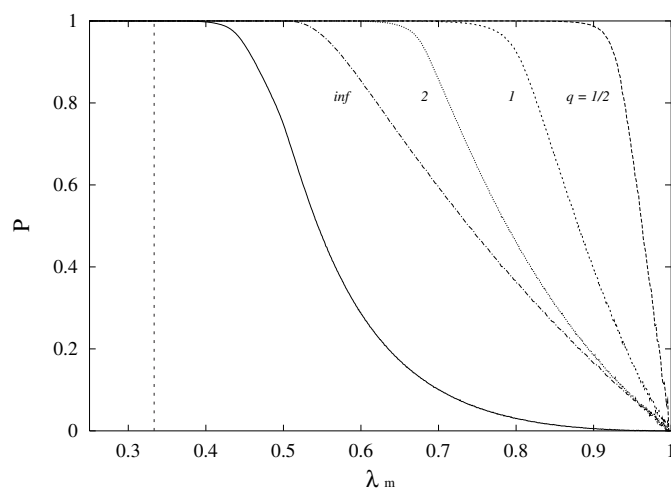
Figure 3. Probability of finding (for different values of q) a two-qubit state which, for a given value of the maximum eigenvalue λ_m, has positive relative q-entropies. The solid line corresponds to the probability of finding, for a given value of λ_m, a two-qubit state with a positive partial transpose.
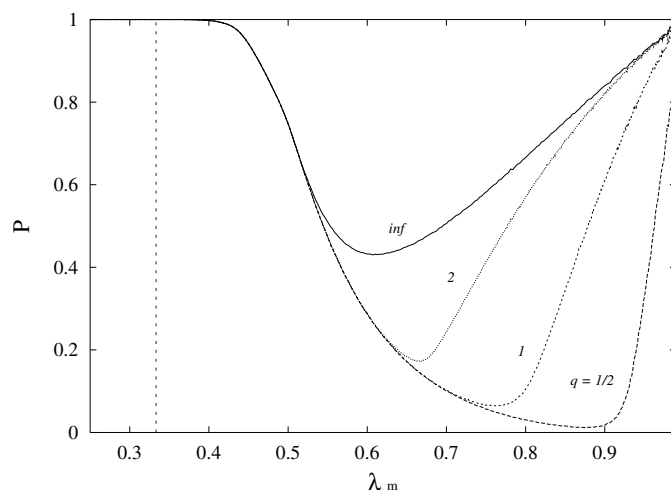

Figure 4. Probability of finding (for different values of q) a two-qubit state which, for a given value of the maximum eigenvalue λ_m, either (i) has its two relative q-entropies positive, as well as a positive partial transpose, or (ii) has a negative relative q-entropy and a non-positive partial transpose.

transpose. We see in figure 3 that, for $\lambda_m < 1/3$, the probability of finding states verifying the classical entropic inequalities (i.e. having positive relative entropies) is, for all $q > 0$, equal to 1. This is so because all states whose largest eigenvalue $\lambda_m \leqslant 1/3$ are separable [8].

Figure 4 depicts the probability of finding (for different values of q) a two-qubit state which, for a given value of the maximum eigenvalue λ_m, either has both (i) relative q-entropies positive and a positive partial transpose, or (ii) a negative relative q-entropy and a non-positive partial transpose.
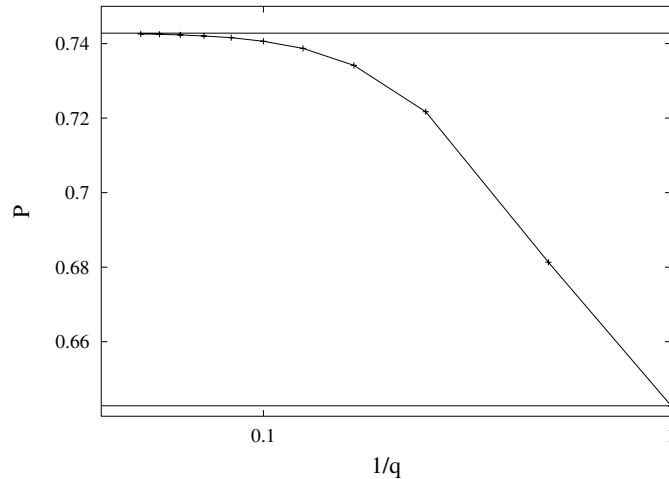
Figure 5. Probability (as a function of q) of finding a two-qubit state which either has both positive relative q-entropies and a positive partial transpose, or has a negative relative q-entropy and a non-positive partial transpose.

A remarkable aspect of the behaviour of the sign of the relative q-entropies, which transpires from figures 1 and 3, is that, for any degree of mixture, the volume corresponding to states with positive relative q-entropies ($q > 0$) is a monotonic decreasing function of q. This feature of figures 1 and 3 is interesting because, for a single given state ρ, the relative q-entropy is not necessarily decreasing in q [27]. This means that the positivity of the relative entropy of a given state ρ and for a given value q^* of the entropic parameter does not imply the positivity of the relative q-entropies of that state for all $q < q*$. That is, $q < q^*$ does not imply that the family of states exhibiting positive relative q^*-entropies is a subset of the family of states with positive q-entropies. This fact notwithstanding, the numerical results reported here indicate that for $0 < q < q*$ the volume of states with positive q^*-relative entropies is smaller than the volume of states with positive q-entropies. This implies that, among all the q-entropic separability criteria, the one corresponding to the limit case $q \to \infty$ is the strongest, as was recently suggested by Abe [34] on the basis of his analysis of a monoparametric family of mixed states for multi-qudit systems.

It is interesting to see the behaviour, as a function of the entropic parameter q, of the global probability (regardless of the degree of mixture) that an arbitrary state of a two-qubit system exhibits simultaneously (i) a positive relative q-entropy and a positive partial transpose, or (ii) a negative relative q-entropy and a non-positive partial transpose. In other words, this is the probability that for an arbitrary state the entropic separability criterion and the Peres criterion lead to the same 'conclusion' with respect to the separability (or not) of the state under consideration. In figure 5 we depict this probability as a function of $1/q$, for values of $q \in [2, 20]$. We see that this probability is an increasing function of q. In the limit $q \to \infty$ this probability approaches the value ≈ 0.7428. On the other hand, for $q = 1$ (that is, when we use the standard logarithmic entropy) the probability is approximately equal to 0.6428.

We have performed for qubit–qutrit systems calculations similar to those which we have already discussed for two-qubit systems. The results are summarized in figures 6 and 7. Figure 6 depicts the probability of finding (for different values of q) a qubit–qutrit state which, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, has positive relative q-entropies. The solid line in figure 6 corresponds to the probability of finding, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, a qubit–qutrit state with a positive partial transpose. Figure 7 depicts
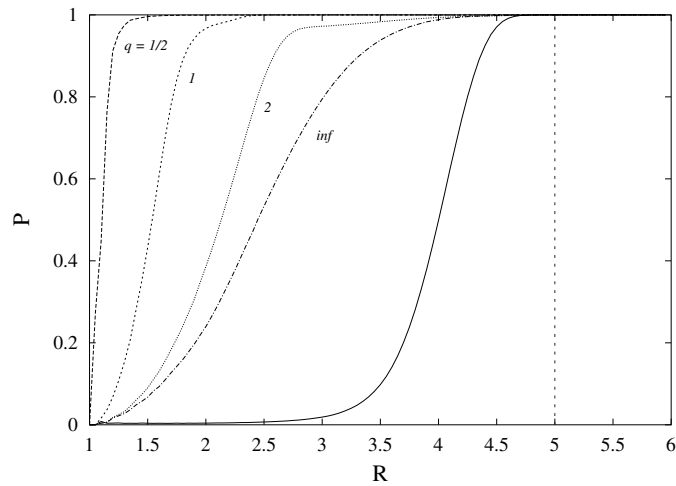
Figure 6. Probability of finding (for different values of q) a qubit–qutrit state which, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, has positive relative q-entropies. The solid line corresponds to the probability of finding, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, a qubit–qutrit state with a positive partial transpose.
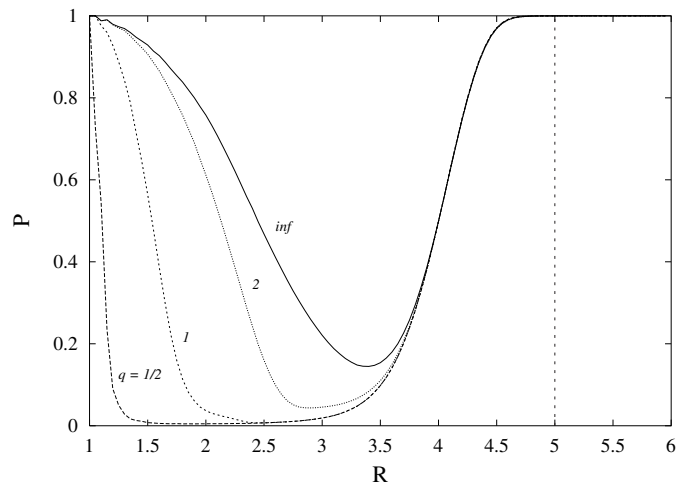

Figure 7. Probability of finding a qubit–qutrit state which, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, and for different values of q, either (i) has its two relative q-entropies positive, as well as a positive partial transpose, or (ii) has a negative relative q-entropy and a non-positive partial transpose.

the probability of finding, for different values of q, a qubit–qutrit state which has, for a given degree of mixture $R = 1/\text{Tr}(\rho^2)$, either (i) its two relative q-entropies positive, as well as a positive partial transpose, or (ii) a negative relative q-entropy and a non-positive partial transpose. We have also computed the probability (for different values of q) that an arbitrary qubit–qutrit state (regardless of its degree of mixture) be correctly classified, either as entangled or as separable, on the basis of the sign of the relative q-entropies. These probabilities are depicted in figure 8, for values of q in the interval $q \in [2, 20]$. As happens with two-qubit
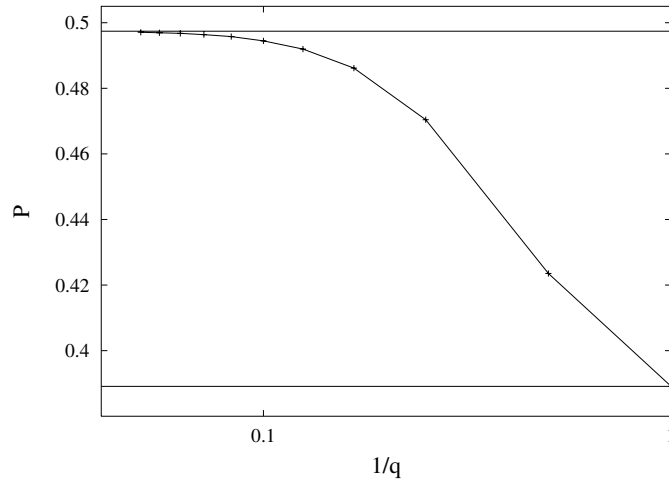
Figure 8. Probability (as a function of q) of finding a qubit–qutrit state which either has both positive relative q-entropies and a positive partial transpose, or has a negative relative q-entropy and a non-positive partial transpose.

systems, this probability is an increasing function of q. For $q = 1$ the probability is approximately equal to 0.3891 and approaches the (approximate) value 0.4974 as $q \to \infty$. For a given value of q, the probability of coincidence between the Peres' and the entropic separability criteria are seen to be smaller in the case of qubit–qutrit systems than in the case of two-qubit systems.

It is worthwhile to investigate the manner in which the (negative) relative q-entropy $S_q^{(R)}(\rho_A) - S_q^{(R)}(\rho_{AB}) = -S_q^{(R)}(B|A)$ is related to the entanglement of formation [42], for general two-qubit states violating the concomitant classical entropic inequality. We have studied the aforementioned relationship numerically. The entanglement of formation of a two-qubit state $\hat{\rho}$ can be evaluated analytically by recourse to Wootters' formula [43],

$$E[\hat{\rho}] = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right) \tag{19}$$

where

$$h(x) = -x \log_2 x - (1 - x) \log_2(1 - x) \tag{20}$$

and the concurrence C is given by

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \tag{21}$$

λ_i ($i = 1, \ldots, 4$) being the square roots, in decreasing order, of the eigenvalues of the matrix $\hat{\rho}\tilde{\rho}$, with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y). \tag{22}$$

The above expression has to be evaluated by recourse to the matrix elements of $\hat{\rho}$ computed with respect to the product basis. In figure 9, the concurrence squared C^2 is plotted versus $S_q^{(R)}(\rho_A) - S_q^{(R)}(\rho_{AB}) = -S_q^{(R)}(B|A)(q = \infty)$, for a set of random two-qubit states generated numerically, keeping only those with a negative relative entropy. It can be appreciated in figure 9 that, for those states not complying with the classical inequality $S_q^{(R)}(B|A) \geqslant 0$, the concurrence squared C^2 (and consequently, the entanglement of formation) is, to a
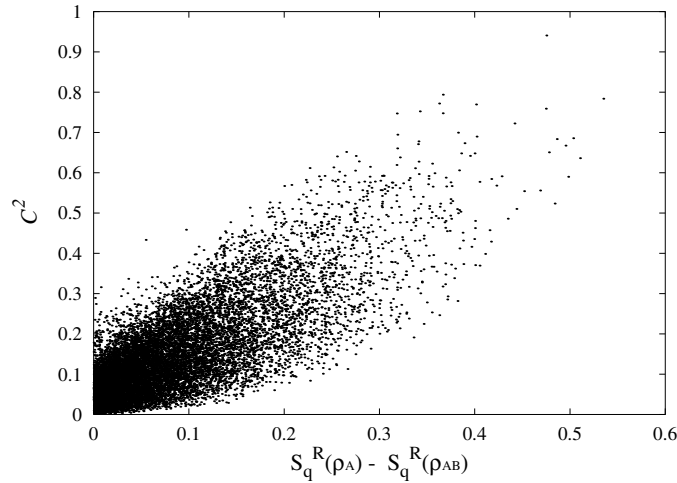
Figure 9. The concurrence squared C^2 is plotted versus $S_q^{(R)}(\rho_A) - S_q^{(R)}(\rho_{AB})(q = \infty)$, for a set of random two-qubit states generated numerically, keeping only those with a negative relative entropy.

certain extent, correlated with the relative q-entropy $S_q^{(R)}(B|A)$. States with increasing (positive) values of $S_q^{(R)}(\rho_A) - S_q^{(R)}(\rho_{AB})$ tend to be endowed with larger values of the concurrence squared C^2. Alas, this correlation does not seem to be strong, and deteriorates as $S_q^{(R)}(\rho_A) - S_q^{(R)}(\rho_{AB})$ increases.

Finally, we have computed the probabilities of finding states with positive relative q-entropies (for the case $q = \infty$) for bipartite quantum systems described by Hilbert spaces of increasing dimensionality. Let N_1 and N_2 stand for the dimensions of the Hilbert spaces associated with each subsystem and $N = N_1 \times N_2$ be the dimension of the Hilbert space associated with the concomitant composite system. We have considered three sets of systems: (i) systems with $N_1 = 2, 3$ and increasing values of N_2 and (ii) systems with $N_1 = N_2$ and increasing dimensionality. The computed probabilities are depicted in figure 10, as a function of the total dimension N. The three upper curves correspond (as indicated in the figure) to composite systems with $N_1 = 2$, $N_1 = 3$ and $N_1 = N_2$. For the sake of comparison, the probability of finding states complying with the Peres partial transpose separability criterion (lower curve) is also plotted. In order to obtain each point in figure 10, 10^8 states were randomly generated.

Some interesting conclusions can be drawn from figure 10. In the case of composite systems with $N_1 = N_2$, the probability of finding states complying with the classical ($q = \infty$) entropic inequalities (that is, having positive both relative q-entropies) is an increasing function of the dimensionality. Furthermore, this probability seems to approach 1, as $N \to \infty$. In other words, figure 10 provides numerical evidence that, in the limit of infinite dimension, typical two-qudit systems (dimension $D \times D$) behave classically, as far as the signs of the relative q-entropies are concerned. This does not mean, of course, that there are no two-qudit states with large D violating the classical inequalities. For any value of D, such states always exist (for instance, entangled pure states). Our numerical findings suggest, however, that for large D, *typical* mixed states of two-qudit systems do comply with the classical entropic inequalities.

When considering composite systems with increasing dimensionality, but keeping the dimension of one of the subsystems constant ($N_1 = 2, 3$), we obtained numerical evidence
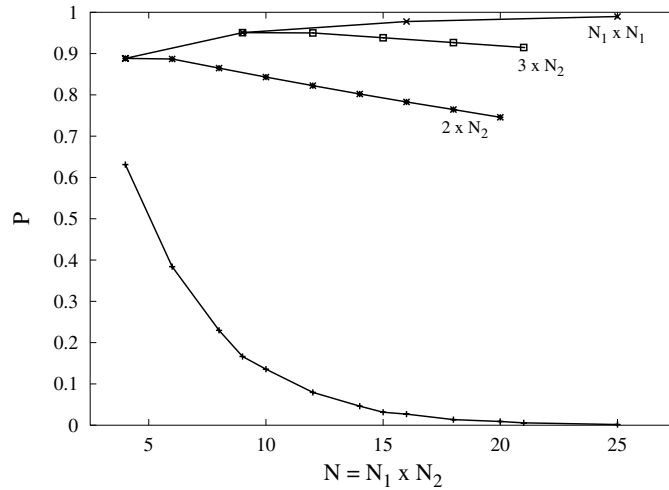
Figure 10. Global probability of finding a state (pure or mixed) of a bipartite quantum system with positive relative q-entropies. N_1 and N_2 stand for the dimensions of the Hilbert spaces associated with each subsystem, and $N = N_1 \times N_2$ is the dimension of the Hilbert space associated with the composite system as a whole. The three upper curves correspond (as indicated in the figure) to composite systems of increasing dimensionality, and with $N_1 = 2$, $N_1 = 3$ and $N_1 = N_2$. The probability of finding a state complying with the Peres partial transpose separability criterion (lower curve) is also plotted.

that the probability of having positive relative q-entropies (again, with $q = \infty$) behaves in a monotonic decreasing way with the total dimension $N = N_1 \times N_2$. As can be appreciated in figure 10, for $N_1 = 2$, and within the range of N-values covered by our numerical computations, the alluded to probability is a linear decreasing function of N. A similar decreasing linear behaviour is exhibited by our results corresponding to $N_1 = 3$ (for large enough N). In this last case, the probability of having positive relative q-entropies (with $q = \infty$) decreases with a slope smaller than the slope associated with $N_1 = 2$. Summing up, our numerical results suggest that the non-classical features of composite systems with $N_1 = $ const, related to the sign of the relative q-entropies, tend to increase with the total dimensionality $N = N_1 \times N_2$.

It is interesting to note that the probabilities of finding states with positive q-entropies are not just a function of the total dimension $N = N_1 \times N_2$ (as happens, with good approximation, for the probability of having a positive partial transpose). In contrast, they depend on the individual dimensions (N_1 and N_2) of both subsystems. Furthermore, the trends of the alluded to probabilities are clearly different if one considers composite systems of increasing dimension with either (i) increasing dimensions for both subsystems or (ii) increasing dimension for one of the subsystems and constant dimension for the other.

4. Conclusions

We have performed a systematic numerical survey of the space of mixed states of two-qubit systems in order to determine, as a function of the degree of mixture, and for different values of the entropic parameter q, the volume in state space occupied by those states characterized by positive values of the relative q-entropy. We also computed, for different values of q, the global probability of correctly classifying an arbitrary state of a two-qubit system (either as separable or as entangled) on the basis of the signs of its relative q-entropies. This probability

exhibits a monotonic increasing behaviour with the entropic parameter q. The approximate values of these probabilities are 0.6428 for $q = 1$ and 0.7428 in the limit $q \to \infty$.

An interesting conclusion that can be drawn from the numerical results reported here is that, notwithstanding the known non-monotonicity in q of the relative q-entropies [27], the volume corresponding to states with positive relative q-entropies ($q > 0$) is, for any degree of mixture, a monotonic decreasing function of q.

Similar calculations were performed for qubit–qutrit systems and for composite systems described by Hilbert spaces of larger dimensionality. We pay particular attention to the limit case $q \to \infty$. Our numerical results indicate that, for composite systems consisting of two subsystems characterized by Hilbert spaces of equal dimension N_1, the probability of finding states with positive q-entropies tends to 1 as N_1 increases. In other words, as $N_1 \to \infty$ most states seem to behave (as far as their relative q-entropies are concerned) classically. On the other hand, the alluded to probability exhibits a decreasing behaviour (as a function of the total dimensionality $N = N_1 \times N_2$) when the dimension N_1 associated with one of the subsystems is kept fixed.

Acknowledgments

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